Rocket efficiency in the presence of drag, v1.0

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1 Goal

We aim to maximize the energy delivered to the payload of the rocket per unit fuel burned. Burn too slowly, and the rocket will simply fight gravity. Burn too quickly, and the rocket will simply fight drag. Our goal is to find the speed at which the rocket should travel through the atmosphere to balance these concerns.

2 Assumptions

- We deal only with rockets travelling straight up. For the period of heaviest drag, this is approximately correct.

- The drag force is proportional to the square of the speed.

3 Definitions

y distance from the center of the planet

v dy/dt

a dv/dt

p mass of payload

m mass of entire vehicle, including payload and fuel

G gravitational constant

M mass of planet

c drag coefficient, where the force due to drag is cv^2

b rate of fuel burn: $-dm/dt$

I specific impulse of rocket engine, where the force exerted by the rocket engine is Ib

4   Efficiency as a function of speed

Total payload energy is the sum of kinetic energy and potential energy:

$$E_p = pv^2 - \frac{GMp}{y} \tag{1}$$

As we burn, we change the mass of the rocket. The change of energy with respect to mass is:

$$\frac{dE_p}{dm} = \frac{dE_p}{dt}\frac{dt}{dm} \tag{2}$$

$$\frac{dE_p}{dm} = -\frac{1}{b}\left(\frac{GMpv}{y^2} + 2pva\right) \tag{3}$$

To simplify the algebra, define $g \equiv \frac{GM}{y^2}$. Then:

$$\frac{dE_p}{dm} = -\frac{1}{b}(gpv + 2pva) \tag{4}$$

The forces on the rocket are gravity, drag, and the rocket engine:

$$ma = -gm - cv^2 + Ib \tag{5}$$

With the rocket at constant speed, net acceleration is zero, so:

$$b = \frac{1}{I}(gm + cv^2) \tag{6}$$

Combining formulas 4 and 6:

$$\frac{dE_p}{dm} = -I\left(\frac{gpv + 2pva}{gm + cv^2}\right) \tag{7}$$

Given that $a = 0$, we have:

$$\frac{dE_p}{dm} = \frac{-Igp}{\frac{gm}{v} + cv} \tag{8}$$

5   Maximizing efficiency

To find the speed which maximizes efficiency, we find the speed such that $\frac{dE_p}{dm}$ is as negative as possible:

$$\frac{d}{dv}\left(\frac{dE_p}{dm}\right) = 0 \tag{9}$$

Substituting formula 8 into formula 9:

$$\frac{Igp}{(\frac{gm}{v} + cv)^2}\left(-\frac{gm}{v^2} + c\right) = 0 \tag{10}$$

$$gm = cv^2 \tag{11}$$

So, we get best efficiency when the force of gravity is equal to the force of drag.